

Painlevé structure of a multi-ion electrodiffusion system

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2007 J. Phys. A: Math. Theor. 40 F1031

(<http://iopscience.iop.org/1751-8121/40/48/F01>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.146

The article was downloaded on 03/06/2010 at 06:27

Please note that [terms and conditions apply](#).

FAST TRACK COMMUNICATION

Painlevé structure of a multi-ion electrodiffusion system**R Conte¹, C Rogers^{2,4} and W K Schief^{3,4}**¹ Service de physique de l'état condensé (URA 2464), CEA–Saclay, F-91191 Gif-sur-Yvette Cedex, France² Department of Applied Mathematics, Hong Kong Polytechnic University, Kowloon, Hong Kong³ Institut für Mathematik, Technische Universität Berlin, Straße des 17. Juni 136, D-10623 Berlin, Germany⁴ Australian Research Council Centre of Excellence for Mathematics and Statistics of Complex Systems, School of Mathematics and Statistics, The University of New South Wales, Sydney, NSW 2052, AustraliaE-mail: Robert.Conte@cea.fr, colinr@maths.unsw.edu.au and schief@math.tu-berlin.de

Received 25 September 2007, in final form 11 October 2007

Published 14 November 2007

Online at stacks.iop.org/JPhysA/40/F1031**Abstract**

A nonlinear coupled system descriptive of multi-ion electrodiffusion is investigated and all parameters for which the system admits a single-valued general solution are isolated. This is achieved via a method initiated by Painlevé with the application of a test due to Kowalevski and Gambier. The solutions can be obtained explicitly in terms of Painlevé transcendents or elliptic functions.

PACS numbers: 02.20.Qs, 11.10.Lm

1. Introduction

The theory of electrodiffusion has its origin in the liquid-junction theory of Nernst and Planck [1]. It seeks to provide a macroscopic description of the transport of charged particles through material barriers. Applications abound, in particular, in the modelling of biological membranes [2–7] and in electrochemistry [8]. Schlögl [9] observed that it is convenient to partition the ions into m classes characterized by the same electric charge $q_j = q_0 v_j$, where q_0 is the unit of charge and v_j is a nonzero integral signed valency. The m -ion electrodiffusion model in steady régimes then reduces to the following system of $m + 1$ coupled first-order ODEs [10]:

$$\begin{cases} \frac{dn_i}{dx} = v_i n_i p - c_i, & v_i n_i \neq 0, \quad i = 1, \dots, m, \\ \frac{dp}{dx} = \sum_{i=1}^m v_i n_i, \end{cases} \quad (1)$$

where x is the coordinate normal to the planar boundaries, p is the electric field and n_j is the number of ions with the same charge $q_j = q_0 v_j$. These variables are subject to the constraint [9]

$$(v_i - v_j)v_i n_i \neq 0, \quad i \neq j. \quad (2)$$

The system (1) admits the first integral

$$K = \frac{p^2}{2} - Cx - \sum_{i=1}^m n_i, \quad (3)$$

where K may be set to zero without loss of generality whenever $C = \sum_{i=1}^m c_i \neq 0$. Note that certain boundary value problems indeed require that K be zero. Here, this constraint is not imposed and it is demonstrated in an algorithmic manner that the electrodiffusion model for $m \leq 4$ admits solutions given in terms of Painlevé transcendents and elliptic functions. The Painlevé reduction obtained in [11] is retrieved as a particular case.

2. Single-valued solutions: a classical procedure

For nonlinear systems such as (1), dependent on certain parameters v_i, c_i , there is a classical method [12] for determining all those parameters for which the general solution of the nonlinear system is single-valued around its movable singularities. An alternative approach due to Kowalevski [13] and Gambier [14] is also available. A detailed account of the latter method may be found in [15] or in the summer school lecture notes [16].

The main steps, to be performed in the sequel, are as follows:

- (i) A scaled version of the nonlinear system (1) is introduced for new fields $N_i(X), P(X)$, via the transformation (this is a particular α -transformation [12]),

$$x = \varepsilon X, \quad p = \varepsilon^{-1} P, \quad n_i = N_i. \quad (4)$$

Thus the limit $\varepsilon \rightarrow 0$ generates

$$\begin{cases} \frac{dN_i}{dX} = v_i N_i P, & v_i \neq 0, & i = 1, \dots, m, \\ \frac{dP}{dX} = \sum_{i=1}^m v_i N_i, \\ 0 = \frac{P^2}{2} - \sum_{i=1}^m N_i. \end{cases} \quad (5)$$

This canonical system results from (1) and (3) by setting $c_i = 0, K = 0$ and is termed the *simplified system*, as opposed to the original *complete system* (1). If the complete system has a single-valued general solution, it is necessary that the simplified system has a single-valued general solution.

- (ii) Necessary conditions on the parameters v_i for the general solution of the simplified system to be single-valued are determined. These conditions are shown to be sufficient for the problem at hand.
- (iii) The necessary conditions on the parameters v_i, c_i for the general solution of the complete system to be single-valued are obtained and the corresponding PII and other reductions listed.

3. Constraints on the charges ν_i

The transformation (4) implies that $P(X)$ admits movable simple poles,

$$P = \chi^{-1}[P_0 + \mathcal{O}(\chi)], \quad \chi = X - X_0, \quad P_0 \neq 0. \quad (6)$$

A straightforward integration of (5)₁ reveals the leading order behaviour

$$\begin{cases} P = \chi^{-1}[P_0 + \mathcal{O}(\chi)], \\ N_i = \chi^{v_i P_0}[A_i + \mathcal{O}(\chi)], \\ 0 = \frac{P_0^2}{2}\chi^{-2}[1 + \mathcal{O}(\chi)] - \sum_{i=1}^m \chi^{v_i P_0}[a_i + \mathcal{O}(\chi)]. \end{cases} \quad (7)$$

Here, the A_i are undetermined constants. Since the v_i are assumed distinct (see (2)), the term in χ^{-2} in (7)₃ can only be matched by exactly one of the terms involving $\chi^{v_i P_0}$. Denoting j as matching index, this yields the desired value of P_0 ,

$$P_0 = -\frac{2}{v_j}, \quad A_j = \frac{2}{v_j^2}. \quad (8)$$

In view of the possible vanishing of the remaining constants A_i , the necessary condition that each $N_i(X)$ be single-valued near $X = X_0$ is not easy to enforce and hence we proceed directly with the other conditions.

An additional necessary condition is that the linearized version of the simplified system near the local behaviour (7) also has the property that its general solution is single-valued near $X = X_0$. Since this linearized system is of Fuchsian type [17, chapter XVI] near $X = X_0$, it is necessary that all its Fuchs indices be integer. This condition is here readily enforced with knowledge of P_0 . Thus, let us introduce the function W by

$$P = -2\frac{W'}{v_j W}. \quad (9)$$

By construction, W has a simple zero near $X = X_0$ (it can be chosen such that $\lim_{X \rightarrow X_0} W/(X - X_0) = 1$). The integration of the simplified system then reduces to the integration of a single ODE for $W(X)$, namely

$$W'^2 - \frac{v_j^2}{2} \sum_{i=1}^m k_i W^{2-2v_i/v_j} = 0, \quad (10)$$

where

$$N_i = k_i W^{-2v_i/v_j}, \quad (11)$$

and the k_i are arbitrary constants. Since $\lim_{X \rightarrow X_0} W/(X - X_0) = 1$, Laurent series in $X - X_0$ are also Laurent series in W . Moreover, since the Fuchs indices and the Painlevé ‘resonances’ are identical [16, section 5.7.1], the values of the Fuchs indices are directly read from (10), and are given by

$$\text{Fuchs indices} = -1, 2 - 2\frac{v_i}{v_j}, \quad i = 1, \dots, m. \quad (12)$$

Finally, since j can be arbitrarily chosen, the diophantine condition to be solved is

$$\forall j \forall i : -2\frac{v_i}{v_j} = \text{integer}. \quad (13)$$

The latter condition is restrictive and, using the shorthand notation $v_1 : v_2 : v_3 = a : b : c$ to indicate

$$\frac{v_1}{a} = \frac{v_2}{b} = \frac{v_3}{c} = \text{arbitrary}, \quad (14)$$

its only distinct solutions are

$$\begin{cases} m = 1 : & \text{no restriction,} \\ m = 2 : & v_1 : v_2 = 1 : -2, \quad 1 : -1, \quad 1 : 2, \\ m = 3 : & v_1 : v_2 : v_3 = 1 : -2 : -1, \quad 1 : -2 : 2, \\ m = 4 : & v_1 : v_2 : v_3 : v_4 = 1 : -2 : -1 : 2, \\ m > 4 : & \text{no solution.} \end{cases} \quad (15)$$

The necessary condition (13) is also proven to be sufficient. Thus, in the first-order ODE (10) for $W(X)$, all the powers of W are integers between 0 and 4 for the values (15) so that this ODE belongs to the binomial type studied by Briot and Bouquet [18, p 58–9] and its general solution $W(X)$ is single-valued. The explicit expressions for P and N_i in terms of W prove the single-valuedness of the general solution of the simplified system (5). This completes the second step as set down in section 2.

Let us remark that the Fuchs indices (12) comprise a subset of $\{-2, -1, 1, 3, 4, 6\}$.

4. Constraints on the c_i

The third step involves a well-known test due to Kowalevski and Gambier. In this context, it proves convenient to eliminate the m variables n_i to obtain the m th order ODE obeyed by the electric field $p(x)$. Thus, taking the derivative of (1)₂ $m - 1$ times, one generates a van der Monde system for the m variables n_i , namely [10]

$$\begin{cases} \sum_{i=1}^m v_i^q n_i = a_q, & q = 1, \dots, m \\ a_1 = p', & pa_q = a'_{q-1} + \sum_{k=1}^m v_k^{q-1} c_k, \quad q = 2, \dots, m. \end{cases} \quad (16)$$

In view of the constraint (2), the solution n_i of this system is unique. Thus, the question of single-valuedness of the general solution (n_i, p) of the complete system (1) reduces to that of p alone. The m th order first-degree ODE for $p(x)$ is defined by the determinant

$$\begin{vmatrix} 1 & \dots & 1 & p^2/2 - (Cx + K) \\ v_1 & \dots & v_m & a_1 \\ \dots & \dots & \dots & \dots \\ v_1^q & \dots & v_m^q & a_q \\ \dots & \dots & \dots & \dots \\ v_1^m & \dots & v_m^m & a_m \end{vmatrix} = 0. \quad (17)$$

For the admissible values of m as set down in (15), one obtains [10]:

$$\begin{cases} m = 1 : & p' + v_1 \left[-\frac{p^2}{2} + Cx + K \right] = 0, \\ m = 2 : & p'' - (v_1 + v_2)pp' + v_1v_2 \left[\frac{p^3}{2} - (Cx + K)p \right] + v_1c_1 + v_2c_2 = 0, \\ m = 3 : & pp''' - p'p'' - (v_1 + v_2 + v_3)p^2p'' + (v_2v_3 + v_3v_1 + v_1v_2)p^3p' \\ & + v_1v_2v_3 \left[-\frac{p^5}{2} + (Cx + K)p^3 \right] - [(v_2 + v_3)v_1c_1 + (v_3 + v_1)v_2c_2 \\ & + (v_1 + v_2)v_3c_3]p^2 - (c_1v_1 + c_2v_2 + c_3v_3)p' = 0, \\ m = 4 : & p^2p'''' - 3pp'p''' + 3p^2p'' - pp''^2 + s_1(-p^3p''' + p^2p'p'') + s_2p^4p'' \\ & - s_3p^5p' + \frac{s_4}{2}p^7 - s_4(Cx + K)p^5 + A_3p^4 + A_1(3p'^2 - pp'') + A_2p^2p' = 0, \end{cases} \quad (18)$$

with the notation

$$\left\{ \begin{array}{l} s_1 = v_1 + v_2 + v_3 + v_4, \\ s_2 = v_1 v_2 + v_1 v_3 + v_1 v_4 + v_2 v_3 + v_2 v_4 + v_3 v_4, \\ s_3 = v_2 v_3 v_4 + v_1 v_3 v_4 + v_1 v_2 v_4 + v_1 v_2 v_3, \\ s_4 = v_1 v_2 v_3 v_4, \\ A_3 = (c_2 + c_3 + c_4)v_2 v_3 v_4 + (c_1 + c_3 + c_4)v_1 v_3 v_4 \\ \quad + (c_1 + c_2 + c_4)v_1 v_2 v_4 + (c_1 + c_2 + c_3)v_1 v_2 v_3, \\ A_2 = (c_1 + c_2)v_1 v_2 + (c_1 + c_3)v_1 v_3 + (c_1 + c_4)v_1 v_4 \\ \quad + (c_2 + c_3)v_2 v_3 + (c_2 + c_4)v_2 v_4 + (c_3 + c_4)v_3 v_4, \\ A_1 = c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4. \end{array} \right. \quad (19)$$

The next set of necessary conditions arises when one investigates the existence of the m Laurent series (one for each chosen j) whose first term is (see (6))

$$p = \sum_{k=0}^{+\infty} p_k (x - x_0)^{k-1}, \quad p_0 = -\frac{2}{v_j}, \quad j = 1, \dots, m. \quad (20)$$

The locations k at which arbitrary constants may appear in the series (20) are identical [16, section 5.7.1] to the Fuchs indices listed in (12).

Whenever k reaches one of the positive Fuchs indices in the list (12), to eliminate the possibility of a movable logarithm in the expansion (20) destructive of the Painlevé property, conditions must be imposed on the c_i , $i = 1, \dots, m$. The results of this classical computation are (appendix):

$$\left\{ \begin{array}{ll} m = 1 : & \text{no additional condition,} \\ m = 2, \quad v_1 : v_2 = 1 : -2 : & c_1 = 0, \quad c_2 = 0, \\ m = 2, \quad v_1 : v_2 = 1 : -1 : & \text{no additional condition,} \\ m = 2, \quad v_1 : v_2 = 1 : 2 : & \text{no additional condition,} \\ m = 3, \quad v_1 : v_2 : v_3 = 1 : -2 : -1 : & c_1 = 0, \quad c_2(2c_2 + 3c_3) = 0, \\ m = 3, \quad v_1 : v_2 : v_3 = 1 : -2 : 2 : & c_1 = 0, \quad c_2 = 0, \\ m = 4, \quad v_1 : v_2 : v_3 : v_4 = 1 : -2 : -1 : 2 : & c_1 = c_2 = c_3 = c_4 = 0. \end{array} \right. \quad (21)$$

The value K of the first integral is unconstrained.

Whenever -2 is a Fuchs index of some family, movable logarithms may also occur but the classical computation recalled in the appendix cannot detect them. Therefore, an analysis up to the perturbation order $n = 6$ has been conducted in the manner described in [19]. All such additional necessary conditions for the absence of such movable logarithmic singularities are identically satisfied.

In order to complete the third step, it remains to examine whether the general solution (n_i, p) of the complete system is indeed single-valued for the cases set down in (21). It proves convenient to make use of one of the m variables w involved in the singular part transformation (see (9)),

$$p = -2 \frac{w'}{v_j w}, \quad (22)$$

namely the one where $|v_j|$ is the greatest of the m charges. To each vanishing c_i , if any, there corresponds a first integral

$$k_i = n_i w^{2v_i/v_j} \neq 0, \quad (23)$$

where the exponent $2v_i/v_j$ is an integer. For each pair (c_i, c_k) with $c_i = c_k = 0$ one can eliminate w between the two associated first integrals to obtain a first integral rational in (n_i, n_k) , namely

$$n_i^{2v_j/v_i} n_k^{-2v_j/v_k} = \text{constant}. \quad (24)$$

5. The m -ion cases: $m = 1, \dots, 4$

5.1. $m = 1$

In this single ion case,

$$n_1 = \frac{p^2}{2} - (Cx + K), \quad p = -\frac{2w'}{v_1 w}, \quad (25)$$

where w satisfies the Airy-type equation [20],

$$w'' - \frac{v_1^2}{2}(Cx + K)w = 0. \quad (26)$$

5.2. $m = 2$

Here, the n_i , $i = 1, 2$ are given by

$$n_1 = \frac{p' - v_1(\frac{p^2}{2} - Cx - K)}{v_2 - v_1}, \quad n_2 = \frac{p' - v_2(\frac{p^2}{2} - Cx - K)}{v_1 - v_2}, \quad (27)$$

and the second-order ODE for $p(x)$ has degree 1 in p'' . All such second-order ODEs which have the Painlevé property have been classified by Painlevé [12] and Gambier [14].

The first case

$$v_1 : v_2 = 1 : -2, \quad c_1 = 0, \quad c_2 = 0, \quad p'' + v_1 p p' - v_1^2 p^3 + 2K v_1^2 p = 0 \quad (28)$$

belongs to class 10 of Gambier and its general solution is elliptic with

$$p = -2\frac{w'}{v_2 w}, \quad n_1 = k_1 w, \quad n_2 = k_2 w^{-2}, \quad \frac{w'^2}{2v_1^2 w^2} - k_1 w - k_2 w^{-2} - K = 0. \quad (29)$$

The second case

$$v_1 : v_2 = 1 : -1, \quad p'' - \frac{1}{2}v_1^2 p^3 + v_1^2(Cx + K) + v_1(c_1 - c_2) = 0 \quad (30)$$

may be reduced to the Painlevé II equation

$$\frac{d^2 U}{dX^2} = 2U^3 + XU + \alpha \quad (31)$$

under the scaling transformation

$$p = \frac{2k}{v_1} U, \quad x = \frac{X}{k} - \frac{K}{C}, \quad k^3 = -v_1^2 C, \quad 2\alpha = \frac{c_1 - c_2}{C}. \quad (32)$$

The third case

$$v_1 : v_2 = 1 : 2, \quad p'' - 3v_1 p p' + v_1^2 p^3 - 2v_1^2(Cx + K) + v_1(c_1 + 2c_2) = 0 \quad (33)$$

belongs to class 5 of Gambier and is linearizable with

$$p = -\frac{2w'}{v_2 w}, \quad w''' - 2v_1^2(Cx + K)w' - v_1(c_1 + 2c_2)w = 0. \quad (34)$$

5.3. $m = 3$

The three cases to be considered are

$$\left. \begin{aligned} (3a): \quad & \nu_1 : \nu_2 : \nu_3 = 1 : -2 : -1, & (c_1, c_2, c_3) &= (0, 0, 1)C, \\ (3b): \quad & \nu_1 : \nu_2 : \nu_3 = 1 : -2 : -1, & (c_1, c_2, c_3) &= (0, 3, -2)C, \\ (3c): \quad & \nu_1 : \nu_2 : \nu_3 = 1 : -2 : 2, & (c_1, c_2, c_3) &= (0, 0, 1)C. \end{aligned} \right\} \quad (35)$$

The class of equations to which the ODE (18)₃ belongs, namely

$$-p^2 p''' + b p p' p'' + d p'^3 + h p^3 p'' + k p^2 p'^2 + q p^4 p' + f p^6 + \text{subdominant terms} = 0, \quad (36)$$

has been investigated in [21–23], but the results therein are insufficient to cover the above three cases. For $C = K = 0$, the integration by elliptic functions (see (10)) was known to Chazy [21]. For arbitrary values of C and K , we proceed directly with the original system (1).

In the first case, two c_i vanish, so that p obeys a second-order ODE and a birational transformation exists between p and w^2 with

$$(3a): \quad \begin{cases} p = -2 \frac{w'}{\nu_2 w}, & n_1 = k_1 w, & n_2 = k_2 w^{-2}, \\ w'' - \frac{w'^2}{2w} + \nu_1^2 \left(-2k_1 w^2 - (Cx + K)w + \frac{k_2}{w} \right) = 0, \\ 3k_2 \nu_1^2 w^{-2} = \frac{p'' - \nu_1^2 p^3 / 2 + \nu_1^2 (Cx + K)p - \nu_1 C}{p}. \end{cases} \quad (37)$$

The ODE for w belongs to class 34 of Gambier and its general solution is an algebraic transform of the PII function U (31) with

$$(3a): \quad \begin{cases} k_1 \nu_1^2 w = \varepsilon k \frac{dU}{dx} + k^2 U^2 - \frac{\nu_1^2}{2} (Cx + K), & x = \frac{X}{k} - \frac{K}{C}, \\ k^3 = -\nu_1^2 C, & 8k_1^2 k_2 = \frac{(2\alpha + \varepsilon)^2 C^2}{\nu_1^2}, & \varepsilon^2 = 1. \end{cases} \quad (38)$$

In the second case, since only one of the c_i vanishes, the ODE for w is of third-order and a birational transformation links p and w with

$$(3b): \quad \begin{cases} p = -2 \frac{w'}{\nu_2 w}, & n_1 = k_1 w, \\ w''' + \nu_1^2 [-6k_1 w w' - 2(Cx + K)w' - 4Cw] = 0, \\ 6k_1 \nu_1^2 w = \frac{p'' + 3\nu_1 p p' + \nu_1^2 p^3 - 2\nu_1^2 (Cx + K)p - 4\nu_1 C}{p}. \end{cases} \quad (39)$$

The ODE for w belongs to class XIII of Chazy [21] and admits the first integral

$$(3b): \quad \begin{cases} K_2 = (w + (Cx + K)k_1^{-1}) w'' - \frac{w'^2}{2} - \frac{Cw'}{k_1} - 2k_1 \nu_1^2 w^3 \\ - 4\nu_1^2 (Cx + K)w^2 - 2\nu_1^2 (Cx + K)^2 \frac{w}{k_1}, \\ = \nu_1 k_1^{-2} n_1 [\nu_1 (Cx + K)n_3 - Cp - \nu_1 n_1 n_2]. \end{cases} \quad (40)$$

This second-order ODE for w again belongs to class 34 of Gambier and it may be integrated in terms of the PII function U with

$$(3b): \quad \begin{cases} k_1 \nu_1^2 w = \varepsilon k \frac{dU}{dx} + k^2 U^2, \\ x = \frac{X}{k} - \frac{K}{C}, & k^3 = 2\nu_1^2 C, & k_1^2 K_2 = -2C^2 \alpha (\alpha + \varepsilon), & \varepsilon^2 = 1. \end{cases} \quad (41)$$

The third case, for which again two c_i vanish, follows the same pattern as the first case and we have

$$(3c): \begin{cases} p = -2 \frac{w'}{v_3 w}, & n_1 = k_1 w^{-1}, & n_2 = k_2 w^2, \\ w'' + v_1^2 [-4k_2 w^3 - 2(Cx + K)w - k_1] = 0, \\ 12k_2 v_1^2 w^2 = \frac{p'' - 3v_1 p p' + v_1^2 p^3 - 2v_1^2 (Cx + K)p + 2v_1 C}{p}, \end{cases} \quad (42)$$

so that a PII equation is obtained for w .

It is remarked that, in the three cases considered above, the first integral of the third-order ODE (18)₃ for p is a polynomial of third degree in p'' . This underlines the advantage in proceeding with w rather than p in our analysis.

5.4. $m = 4$

In the unique case to be integrated,

$$v_1 : v_2 : v_3 : v_4 = 1 : -2 : -1 : 2, \quad c_1 = c_2 = c_3 = c_4 = 0, \quad (43)$$

the four first integrals immediately yield the solution

$$\begin{cases} p = -\frac{w'}{2v_2 w}, & n_1 = k_1 w, & n_2 = k_2 w^{-2}, & n_3 = k_3 w^{-1}, & n_4 = k_4 w^2, \\ \frac{w'^2}{2v_1^2 w^2} - k_1 w - k_2 w^{-2} - k_3 w^{-1} - k_4 w^2 - K = 0, \end{cases} \quad (44)$$

so that w in general is an elliptic function.

The case when m is arbitrary and all the c_i 's vanish is similarly reduced to a first-order ODE integrable by a quadrature, however with a multi-valued solution when $m > 4$.

6. Conclusion

The analysis presented here has been used to isolate, in particular, underlying Painlevé II structure in the 3-ion electrodiffusion model. This has potential application to the construction of upper and lower solutions to two-point boundary value problems [24]. The integrable nature of the model in these cases also allows the application of Bäcklund transformations [25].

Acknowledgments

Robert Conte warmly thanks the Australian Research Council Centre of Excellence for Mathematics and Statistics of Complex Systems at the University of New South Wales for support for the visit in which this work was started.

Appendix. Conditions for the absence of movable logarithms

After the coefficient p_0 of the series (20) has been computed, the recurrence relation for subsequent p_k , namely

$$\forall k \geq 1 : E_k \equiv P(k)p_k + Q_k(\{p_l | l < k\}) = 0, \quad (A.1)$$

depends linearly on p_k and nonlinearly on the previously computed coefficients p_l . Whenever the positive integer k is a Fuchs index r of the m th order ODE (17), the coefficient $P(r)$

vanishes and the condition $Q_r = 0$ must be enforced in order to avoid movable logarithms. For the cases isolated in (15) and for each of the m families $p_0 = -2/v_j$, these conditions $Q_r = 0$ deliver the following:

$m = 2$:

$$\left\{ \begin{array}{ll} v_1 : v_2 = 1 : -2, & j = 1, & Q_6 = c_2(c_1 + 2c_2), \\ v_1 : v_2 = 1 : -2, & j = 2, & Q_3 = c_1, \\ v_1 : v_2 = 1 : -1, & j = 1, & Q_4 = 0, \\ v_1 : v_2 = 1 : -1, & j = 2, & Q_4 = 0, \\ v_1 : v_2 = 1 : 2, & j = 1, & \text{no condition,} \\ v_1 : v_2 = 1 : 2, & j = 2, & Q_1 = 0. \end{array} \right. \quad (\text{A.2})$$

$m = 3$:

$$\left\{ \begin{array}{lll} v_1 : v_2 : v_3 = 1 : -2 : -1, & j = 1, & Q_4 = 0, & Q_6 = c_2(c_1 + 2c_2 + 3c_3), \\ v_1 : v_2 : v_3 = 1 : -2 : -1, & j = 2, & Q_1 = 0, & Q_3 = c_1, \\ v_1 : v_2 : v_3 = 1 : -2 : -1, & j = 3, & Q_4 = 0, & \\ v_1 : v_2 : v_3 = 1 : -2 : 2, & j = 1, & Q_6 = c_2(5c_1 + 10c_2 + 6c_3), & \\ v_1 : v_2 : v_3 = 1 : -2 : 2, & j = 2, & Q_3 = c_1, & Q_4 = 0, \\ v_1 : v_2 : v_3 = 1 : -2 : 2, & j = 3, & Q_1 = 0, & Q_4 = c_2 p_1. \end{array} \right. \quad (\text{A.3})$$

$m = 4$: $v_1 : v_2 : v_3 : v_4 = 1 : -2 : -1 : 2$,

$$\left\{ \begin{array}{ll} j = 1, & Q_4 = 0, & Q_6 = c_2(5c_1 + 10c_2 + 15c_3 + 6c_4), \\ j = 2, & Q_1 = 0, & Q_3 = c_1, & Q_4 = (3c_1 + 2c_4)p_1, \\ j = 3, & Q_4 = 0, & Q_6 = c_4(5c_3 + 10c_4 + 15c_1 + 6c_2), \\ j = 4, & Q_1 = 0, & Q_3 = c_3, & Q_4 = (3c_3 + 2c_2)p_1. \end{array} \right. \quad (\text{A.4})$$

Since the coefficient p_1 must remain arbitrary, this leads to the list (21).

References

- [1] Planck M 1890 *Ann. Phys. Chem.* **39** 161
- [2] Cole K S 1986 *Membranes, Ions and Impulses* (Berkeley, CA: University of California Press)
- [3] Schwarz T L 1971 *Biophysics and Physiology of Excitable Membranes* ed W J Adelman Jr (Princeton, NJ: Van Nostrand-Reinhold)
- [4] Sandblom J 1972 *Biophys. J.* **12** 1118
- [5] Hagglund J V 1972 *J. Membr. Biol.* **10** 153
- [6] Mackey M C 1975 *Math. Biosci.* **25** 67
- [7] Leuchtag H R and Swihart J C 1977 *Biophys. J.* **17** 27
- [8] Bockris J O'M and Reddy A K N 1970 *Modern Electrochemistry* (New York: Plenum)
- [9] Schlögl R 1954 *Z. Phys. Chem.* **1** 305
- [10] Leuchtag H R 1981 *J. Math. Phys.* **22** 1317
- [11] Bass L 1964 *Trans. Faraday Soc.* **60** 1656
- [12] Painlevé P 1900 *Bull. Soc. Math. France* **28** 201
- [13] Kowalewski S 1889 *Acta Math.* **12** 177
- [14] Gambier B 1910 *Acta Math.* **33** 1
- [15] Bureau F J 1964 *Annali di Mat. Pura ed applicata* **64** 229
- [16] Conte R 1999 *The Painlevé Property, One Century later (CRM Series in Mathematical Physics vol 77)* ed R Conte (New York: Springer)
- [17] Ince E L 1926 *Ordinary Differential Equations* (London: Longmans Green)
- Ince E L 1939 *Ordinary Differential Equations* (Khar'kov: GTIU) (Russian translation)
- Ince E L 1956 *Ordinary Differential Equations* (New York: Dover) (reprinted)
- [18] Painlevé P 1897 *Leçons sur la théorie analytique des équations différentielles (Leçons de Stockholm, 1895)* (Paris: Hermann)

- Painlevé P 1973 *Leçons sur la théorie analytique des équations différentielles (Œuvres de Paul Painlevé vol I)* (Paris: Éditions du CNRS) (reprinted)
- [19] Conte R, Fordy A P and Pickering A 1993 *Physica D* **69** 33
- [20] Fan H Y 1948 *Phys. Rev.* **74** 1505
- [21] Chazy J 1911 *Acta. Math.* **34** 317
- [22] Kozulin and Lukashevich N A 1988 *Differentsial'nye Uravneniya* **24** 2064
- [23] Kessi A and Adjabi Y 2004 Third order differential equations with fixed critical points, private communication
- [24] Amster P, Mariani M C, Rogers C and Tisdell C C 2004 *J. Math. Anal. Appl.* **289** 712
- [25] Rogers C, Bassom A P and Schief W K 1999 *J. Math. Anal. Appl.* **240** 367